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Finite element analysis of the thermoelastic interactions in an unbounded body with a cavity

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Abstract Thermoelastic interactions in an infinite homogeneous elastic medium with a spherical or cylindrical cavity are studied. The cavity surface is subjected to a ramp-type heating of its internal boundary which is assumed to be traction free. A general finite element model is proposed to analyze transient phenomena in thermoelastic solids. Lord–Shulman and Green–Lindsay for the generalized thermoelasticity model are selected for that purpose since it allows for "second sound" effects and reduces to the classical model by appropriate choice of the parameters. The problem has been solved numerically using a finite element method (FEM). Numerical results for the temperature distribution, displacement, radial stress and hoop stress are represented graphically. A comparison is made with the results predicted by the three theories.

List of symbols

Lame's constants	$\lambda,\ \mu$
Density of the medium	ρ
Specific heat at constant strain	C_v
Coefficient of linear thermal expansion	α_t
Time	t
Temperature	T
Reference temperature	T_o
Thermal conductivity	K
Heat source	Q
Relaxation times	$t_1,\ t_2,\ t_3$
Time of rise of temperature	t_o
Kronecker symbol	δ_{ij}
Domain	Γ
The weighting functions	$\delta u,\ \delta T$
Components of stress tensor	τ_{ij}
Components of displacement vector	u_i
Body force vector	F_i

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1 Introduction

The generalized theory of thermoelasticity has drawn widespread attention because it removes the physically unacceptable situation of the classical theory of thermoelasticity, that is, that the thermal disturbance propagates with infinite velocity. It is receiving serious attention of different researchers. Because of the advancement of pulsed lasers, fast burst nuclear reactors and particle accelerators, etc. which can supply heat pulses with a very fast time-rise. Lord and Shulman [1] introduced the theory of generalized thermoelasticity with one relaxation time for the special case of an isotropic body. The temperature-rate dependent theory of thermoelasticty developed by Green and Lindsay [2], which is also known as thermoelasticity theory with two relaxation times. Lord–Shulman and Green–Lindsay are two important models of generalized theory of thermoelasticity. According to these generalized theories, heat propagation can be visualized as a wave phenomenon rather than a diffusion one; in the literature, it is usually referred to as the second sound effect. Potential applications for these new theories have been reported in the survey article by Ignaczak [3] and references therein. Bahar and Hetnarski [4, 5] developed a method for solving coupled thermoelastic problems by using the state-space approach in which the problem is rewritten in terms of the state-space variables, namely the temperature, the displacement, and their gradients. During the last three decades a number of investigations [6–9] have been carried out using the aforesaid theories of generalized thermoelasticity. Chandrasekharaiah [10] studied one-dimensional waves in a homogeneous isotropic half-space due to sudden inputs of temperature and stress/strain on the boundary by employing the Laplace transform method in the context of thermoelasticity without energy dissipation. In a paper by Abd-alla and Abbas [11], a problem of generalized magnetothermoelasticity for an infinitely long, perfectly conducting cylinder has been studied. Note that in most of the earlier studies, mechanical or thermal loading on the bounding surface was considered to be in the form of a shock. However, the sudden jump of the load is merely an idealized situation because it is impossible to realize a pulse described mathematically by a step function; even a very rapid rise time (of the order 10^{-9} s) may be slow in terms of the continuum. This is particularly true in the case of second sound effects when the thermal relaxation times for typical metals are less than 10^{-9} s. It is thus felt that a finite rise time of external load (mechanical or thermal) applied on the surface should be considered while studying a practical problem of this nature. Considering this aspect of rise time, Misra et al. [12–14] and Youssef [15] solved some problems involving a ramp-type heating at the bounding surface. Chandrasekharaiah and Keshavan [16] and Choudhury [17] studied axisymmetric thermoelastic interactions in an unbounded body with cylindrical cavity.

The exact solution of the generalized thermoelasticity theory governing equations for a coupled and nonlinear/linear exists only for very special and simple initial- and boundary problem. In view of calculating general problems, a numerical solution technique is to be used. For this reason the finite element method is chosen. The method of weighted residuals offers us the formulation of the finite element equations and we obtain a best approximated solutions to linear and nonlinear ordinary and partial differential equations. Applying this method basically involves three steps. The first step is to assume the general behavior of the unknown field variables in such a way as satisfy the given differential equations. Substitution of these approximating functions into the differential equations and boundary conditions result in some errors, called the residual. This residual has to vanish in an average sense over the solution domain. The second step is the time integration. The time derivatives of the unknown variables have to be determined by former results. The third step is to solve the equations resulting from the first and the second step by the solving algorithm of the finite element program (see Zienkiewicz [18]).

The object of the present paper is to study the numerical solution of thermoelastic problem in an unbounded body with a spherical or cylindrical cavity subjected to a ramp-type heating applied to the boundary of the cavity. The problem has been solved numerically using a finite element method (FEM). Numerical results for the temperature distribution, displacement, radial stress and hoop stress are represented graphically. A comparison is made with the results predicted by the three theories.

2 Formulation of the problem

We consider an unbounded homogeneous and isotropic elastic medium which possesses a spherical or cylindrical cavity of radius r_1. In the context of generalized thermoelasticity theories, the system of equations that include the displacement, the stress, the strain and the temperature for a linear, homogenous and isotropic thermoelastic continuum take the following form [14]

$$(\lambda+\mu)u_{j,\,ij}+\mu u_{i,\,jj}+F_i-\gamma\left(T_{,\,i}+t_1\dot{T}_{,\,i}\right)=\rho\ddot{u}_i, \tag{1}$$

the energy equation has the form

$$KT_{,\,ii}=\rho C_v\left(\dot{T}+t_2\ddot{T}\right)+\left(1+t_3\frac{\partial}{\partial t}\right)\left(\gamma T_o\dot{u}_{i,\,i}-\rho Q\right), \tag{2}$$

the constitutive equations are given by

$$\tau_{ij}=\lambda u_{i,\,i}\delta_{ij}+\mu\left(u_{i,\,j}+u_{j,\,i}\right)-\gamma\left(T+t_1\dot{T}\right)\delta_{ij}\,. \tag{3}$$

The above equations reduce to equations of classical dynamical coupled theory (CD), Lord and Shulman's theory (LS) and Green and Lindsay's theory (GL) as:

i. Classical dynamical coupled theory (CD, 1956)

$$t_1=0,\ t_2=0,\ t_3=0\,.$$

ii. Lord and Shulman's theory (LS, 1967)

$$t_1=0,\ t_2=t_3>0\,.$$

iii. Green and Lindsay's theory (GL, 1972)

$$t_1>0,\ t_2>0,\ t_3=0\,.$$

Case 1:

Let us consider a perfectly conducting elastic unbounded body with cylindrical cavity occupying the region $r_1\le r<\infty$ of an isotropic homogeneous medium whose state can be expressed in terms of the space variable $r$and the time variable t. We use a cylindrical system of coordinates $(r,\ \theta,\ z)$,for axially symmetric problem, we have $u_r=u_r(r,\ z,\ t)$, $u_\theta=0$, $u_z=u_z(r,\ z,\ t)$. Furthermore, if only axisymmetric plane strain problem is considered, we have

$u_r = u_r(r,\ t)$ and $u_\theta = u_z = 0$. Therefore, the radial strain e_{rr} and the hoop strain $e_{\theta\theta}$ are given by:

$$e_{rr} = \frac{\partial u}{\partial r}\ ,\ \ e_{\theta\theta} = \frac{u}{r}, \tag{4}$$

where u is a radial displacement. The stress-displacement relations are:

$$\begin{aligned}\tau_{rr} &= (\lambda + 2\mu)\,\tfrac{\partial u}{\partial r} + \lambda\tfrac{u}{r} - \gamma\left(T + t_1\tfrac{\partial T}{\partial t}\right),\\ \tau_{\theta\theta} &= \lambda\tfrac{\partial u}{\partial r} + (\lambda + 2\mu)\,\tfrac{u}{r} - \gamma\left(T + t_1\tfrac{\partial T}{\partial t}\right).\end{aligned} \tag{5}$$

It is assumed that there are no body forces and heat sources in the medium and that the plane, the equation of motion and energy equation have the form:

$$(\lambda + 2\mu)\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right] - \gamma\frac{\partial}{\partial r}\left(T + t_1\frac{\partial T}{\partial t}\right) = \rho\frac{\partial^2 u}{\partial t^2}, \tag{6}$$

$$\begin{aligned}\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} &= \frac{\rho C_v}{K}\left(\frac{\partial}{\partial t} + t_2\frac{\partial^2}{\partial t^2}\right)T\\ &\quad + \frac{T_o\gamma}{K}\left(\frac{\partial}{\partial t} + t_3\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right).\end{aligned} \tag{7}$$

Case 2:

In this case, we assume that the center of the cavity is at the origin of the spherical polar coordinates. The resulting thermoelastic interactions are spherically symmetric and the only non-zero displacement is $u = u(r,\ t)$. Then the non-zero stress components are

$$\begin{aligned}\tau_{rr} &= (\lambda + 2\mu)\,\tfrac{\partial u}{\partial r} + \lambda\tfrac{2u}{r} - \gamma\left(T + t_1\tfrac{\partial T}{\partial t}\right),\\ \tau_{\theta\theta} &= \lambda\tfrac{\partial u}{\partial r} + (\lambda + \mu)\,\tfrac{2u}{r} - \gamma\left(T + t_1\tfrac{\partial T}{\partial t}\right).\end{aligned} \tag{8}$$

The equation of motion in the absence of body forces is given by

$$(\lambda + 2\mu)\left[\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2u}{r^2}\right] - \gamma\frac{\partial}{\partial r}\left(T + t_1\frac{\partial T}{\partial t}\right) = \rho\frac{\partial^2 u}{\partial t^2}. \tag{9}$$

The energy equation without heat sources has the form

$$\begin{aligned}\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} &= \frac{\rho C_v}{K}\left(\frac{\partial}{\partial t} + t_2\frac{\partial^2}{\partial t^2}\right)T\\ &\quad + \frac{T_o\gamma}{K}\left(\frac{\partial}{\partial t} + t_3\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right).\end{aligned} \tag{10}$$

The Eqs. 5–10 reduce to the equations

$$\begin{aligned}\tau_{rr} &= (\lambda + 2\mu)\,\tfrac{\partial u}{\partial r} + n\lambda\tfrac{u}{r} - \gamma\left(T + t_1\tfrac{\partial T}{\partial t}\right),\\ \tau_{\theta\theta} &= \lambda\tfrac{\partial u}{\partial r} + (n\lambda + 2\mu)\,\tfrac{u}{r} - \gamma\left(T + t_1\tfrac{\partial T}{\partial t}\right),\end{aligned} \tag{11}$$

$$\begin{aligned}(\lambda + 2\mu)\left[\frac{\partial^2 u}{\partial r^2} + n\left(\frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right)\right] - \gamma\frac{\partial}{\partial r}\left(T + t_1\frac{\partial T}{\partial t}\right)\\ = \rho\frac{\partial^2 u}{\partial t^2},\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{\partial^2 T}{\partial r^2} + \frac{n}{r}\frac{\partial T}{\partial r} &= \frac{\rho C_v}{K}\left(\frac{\partial}{\partial t} + t_2\frac{\partial^2}{\partial t^2}\right)T\\ &\quad + \frac{T_o\gamma}{K}\left(\frac{\partial}{\partial t} + t_3\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial u}{\partial r} + \frac{nu}{r}\right).\end{aligned} \tag{13}$$

With $n = 1$ give the first case and $n = 2$ for the second case. Then according to our assumption the initial and boundary conditions are

$$u(r,\ 0) = \frac{\partial u(r,\ 0)}{\partial t} = 0\ ,\quad T(r,\ 0) = \frac{\partial T(r,\ 0)}{\partial t} = 0\,, \tag{14}$$

$$\tau_{rr}(r_1,\ t) = 0\ ,\quad T(r_1,\ t) = \begin{cases} 0 & t \le 0\\ T_1\frac{t}{t_o} & 0 < t \le t_o\\ T_1 & t > t_o\end{cases} \tag{15}$$

T_1 being a constant. It is convenient to change the preceding equations into the dimensionless forms. To do this, the dimensionless parameters are introduced as

$$(r^\circ,\ r_1^\circ,\ u^\circ) = \frac{c}{\chi}(r,\ r_1,\ u)\ ,\quad (T^\circ,\ T_1^\circ) = \frac{1}{T_o}(T,\ T_1), \tag{16}$$

$$\begin{aligned}(t^\circ,\ t_o^\circ,\ t_1^\circ,\ t_2^\circ,\ t_3^\circ) &= \frac{c^2}{\chi}(t,\ t_o,\ t_1,\ t_2,\ t_3),\ \left(\tau_{rr}^\circ,\ \tau_{\theta\theta}^\circ\right)\\ &= \frac{1}{\mu}\ (\tau_{rr},\ \tau_{\theta\theta})\ ,\end{aligned} \tag{17}$$

with $c^2 = \frac{\lambda+2\mu}{\rho}$ and $\chi = \frac{K}{\rho C_v}$. Substituting into Eqs. 11–15, one may obtain (after dropping the superscript° for convenience)

$$\begin{aligned}\begin{bmatrix}\tau_{rr}\\ \tau_{\theta\theta}\end{bmatrix} &= \begin{bmatrix}\beta^2\\ \beta^2 - 2\end{bmatrix}\frac{\partial u}{\partial r} + \begin{bmatrix} n\left(\beta^2 - 2\right)\\ n\left(\beta^2 - n + 1\right)\end{bmatrix}\\ &\quad \times \frac{u}{r} - \beta^2\xi\left(T + t_1\frac{\partial T}{\partial t}\right)\begin{bmatrix}1\\ 1\end{bmatrix},\end{aligned} \tag{18}$$

$$\frac{\partial^2 u}{\partial r^2} + n\left(\frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) - \xi\frac{\partial}{\partial r}\left(T + t_1\frac{\partial T}{\partial t}\right) = \frac{\partial^2 u}{\partial t^2}, \tag{19}$$